

Noncommutative Topological Theories of Gravity

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Abstract

The possibility of noncommutative topological gravity arising in the same manner as Yang-Mills theory is explored. We use the Seiberg-Witten map to construct such a theory based on a $SL(2, \mathbb{C})$ complex connection, from which the Euler characteristic and the signature invariant are obtained. This gives us a way towards the description of noncommutative gravitational instantons as well as noncommutative local gravitational anomalies.

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1. INTRODUCTION

The idea of the noncommutative nature of space-time coordinates is quite old [1]. Many authors have extensively studied it from the mathematical [2], as well as field theoretical points of view (for a review, see for instance [3, 4]).

Recently, noncommutative gauge theory has attracted a lot of attention, specially in connection with M(atrix) [5] and string theory [6]. In particular, Seiberg and Witten [6] have found noncommutativity in the description of the low energy excitations of open strings (possibly attached to D-branes) in the presence of a NS constant background B -field. Moreover, they have observed that, depending on the regularization scheme of the two dimensional correlation functions, Pauli-Villars or point splitting, ordinary and noncommutative gauge fields can be induced from the same worldsheet action. Thus, the independence of the regularization scheme tells us that there is a relation of the resulting theory of noncommutative gauge fields, deformed by the Moyal star-product, or Kontsevich star product for systems with general covariance, with a gauge theory in terms of usual commutative fields. This relation is the so-called Seiberg-Witten map.

In string theory, gravity and gauge theories are realized in very different ways. The gravitational interaction is associated with a massless mode of closed strings, while Yang-Mills theories are more naturally described in open strings or in heterotic string theory. Furthermore, as mentioned, string theory predicts a noncommutative effective Yang-Mills theory. Thus the question emerges, whether a noncommutative description of gravity would arise from it. This is a difficult question and it will not be addressed here. However, in a recent paper [7], gravitation on noncommutative D-branes has been discussed.

In this context, recently Chamseddine has made several proposals for noncommutative formulations of Einstein's gravity [8, 9, 10], where a Moyal deformation is done. Moreover, in [9, 10], he gives a Seiberg-Witten map for the vierbein and the Lorentz connection, which is obtained starting from the gauge transformations, of $SO(4,1)$ in the first work, and of $U(2,2)$ in the second one. However, in both cases the actions are not invariant under the full noncommutative transformations. Namely, in [9] the action does not have a definite noncommutative symmetry, and in [10] the Seiberg-Witten map is obtained for $U(2,2)$, but the action is invariant under the subgroup $U(1,1) \times U(1,1)$. These actions deformed by the Moyal product, with a constant noncommutativity parameter, are not

diffeomorphism invariant. However, as pointed out in these works, [9, 10], they could be made diffeomorphism invariant, substituting the Moyal \ast_M -product by the Kontsevich \ast_K -product. For other recent proposals of noncommutative gravity, see [11, 12, 13, 14, 15, 16].

Further, as shown in [17, 18, 19, 20, 21], starting from the Seiberg-Witten map, noncommutative gauge theories with matter fields based on any gauge group can be constructed. In this way, a proposal for the noncommutative standard model based on the gauge group product $SU(3) \times SU(2) \times U(1)$ has been constructed [22]. In these developments, the key argument is that no additional degrees of freedom have to be introduced in order to formulate noncommutative gauge theories. That is, although the explicit symmetry of the noncommutative action corresponds to the enveloping algebra of the limiting commutative symmetry group, it is also invariant with respect to this commutative group, fact made manifest by the Seiberg-Witten map.

In this paper, following these results, we present a first step towards a noncommutative theory of gravity in four dimensions, fully symmetric under the noncommutative symmetry.

We make a proposal for a noncommutative topological quadratic theory of gravity from which the corresponding topological invariants of Riemannian manifolds, the Euler characteristic and the signature, can be obtained. These invariants should classify gravitational instantons. Further, in this context of noncommutative gravity, we explore the notion of gravitational instanton. Other possible global aspects of noncommutative gravity like gravitational anomalies will be briefly addressed as well.

The paper is organized as follows. In section 2 we quickly review the noncommutative gauge theories. In section 3 the main features of topological quadratic gravity are introduced, for the $SO(3,1)$ gauge group, by means of the complex formulation based on the self-dual topological quadratic gravity. In section 4 we present noncommutative topological gravity, with explicit results up to order θ^3 . In section 5, based in the study of the global properties of the noncommutative version of the Lorentz and diffeomorphism groups, we explore the possibility of a definition of noncommutative gravitational instantons, as well as local gravitational anomalies for a theory of gravity. Finally, section 6 contains our conclusions.

2. NONCOMMUTATIVE GAUGE SYMMETRY AND THE SEIBERG-WITTEN MAP

We start this section with conventions and properties of noncommutative spaces for future reference. For a recent review see e.g. [23]. Noncommutative spaces can be understood as generalizations of the usual quantum mechanical commutation relations, by the introduction of a linear operator algebra \mathcal{A} , with a noncommutative associative product,

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}, \quad (1)$$

where \hat{x}^μ are linear operators acting on the Hilbert space $L^2(\mathbf{R}^n)$ and $\theta^{\mu\nu} = -\theta^{\nu\mu}$ are real numbers. The Weyl-Wigner-Moyal correspondence establishes (under certain conditions) an isomorphic relation between \mathcal{A} and the algebra of functions on \mathbf{R}^n , the last with an associative and noncommutative \star -product, the Moyal product, given by

$$f(x) \star g(x) \equiv \left[\exp \left(\frac{i}{2} \theta^{\mu\nu} \frac{\partial}{\partial \varepsilon^\mu} \frac{\partial}{\partial \eta^\nu} \right) f(x + \varepsilon) g(x + \eta) \right]_{\varepsilon=\eta=0}. \quad (2)$$

In order to avoid causality problems we will take $\theta^{0\nu} = 0$.

Due to the fact that we will be working with nonabelian groups, we must include also matrix multiplication, so a $*$ -product will be used as the matrix multiplication with \star -product. Inside integrals, this product has the property $\text{Tr} \int f_1 * f_2 * f_3 * \cdots * f_n = \text{Tr} \int f_n * f_1 * f_2 * f_3 * \cdots * f_{n-1}$. In particular, the trace of the integral of the product of two functions has the property that $\text{Tr} \int f_1 * f_2 = \text{Tr} \int f_1 f_2$.

Let us consider a gauge theory with a hermitian connection, invariant under a symmetry Lie group G , with gauge fields A_μ ,

$$\delta_\lambda A_\mu = \partial_\mu \lambda + i [\lambda, A_\mu], \quad (3)$$

where $\lambda = \lambda^i T_i$, and T_i are the generators of the Lie algebra \mathcal{G} of the group G , in the adjoint representation. These transformations are generalized for the noncommutative theory as,

$$\delta_\lambda \hat{A}_\mu = \partial_\mu \hat{\Lambda} + i \left[\hat{\Lambda} \star \hat{A}_\mu \right], \quad (4)$$

where the noncommutative parameters $\hat{\Lambda}$ have some dependence on λ and the connection A . The commutators $[A \star B] \equiv A * B - B * A$ have the correct derivative properties when acting on products of noncommutative fields.

Due to noncommutativity, commutators like $[\hat{\Lambda} *, \hat{A}_\mu]$ take values in the enveloping algebra of \mathcal{G} in the adjoint representation, $\mathcal{U}(\mathcal{G}, \text{ad})$. Therefore, $\hat{\Lambda}$ and the gauge fields \hat{A}_μ will also take values in this algebra. In general, for some representation R , we will denote $\mathcal{U}(\mathcal{G}, R)$ the corresponding section of the enveloping algebra $\mathcal{U}(\mathcal{G})$. Let us write for instance $\hat{\Lambda} = \hat{\Lambda}^I T_I$ and $\hat{A} = \hat{A}^I T_I$, then,

$$[\hat{\Lambda} *, \hat{A}_\mu] = \left\{ \hat{\Lambda}^I *, \hat{A}_\mu^J \right\} [T_I, T_J] + [\hat{\Lambda}^I *, \hat{A}_\mu^J] \{T_I, T_J\}, \quad (5)$$

where $\{A *, B\} \equiv A * B + B * A$ is the noncommutative anticommutator. Thus all the products of the generators T_I will be needed in order to close the algebra $\mathcal{U}(\mathcal{G}, \text{ad})$. Its structure can be obtained by successive computation of commutators and anticommutators starting from the generators of \mathcal{G} , until it closes,

$$[T_I, T_J] = i f_{IJ}{}^K T_K, \quad \{T^I, T^J\} = d_{IJ}{}^K T_K.$$

The field strength is defined as $\hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu - i[\hat{A}_\mu *, \hat{A}_\nu]$, hence it takes also values in $\mathcal{U}(\mathcal{G}, \text{ad})$. From Eq. (4) it turns out that,

$$\delta_\lambda \hat{F}_{\mu\nu} = i \left(\hat{\Lambda} * \hat{F}_{\mu\nu} - \hat{F}_{\mu\nu} * \hat{\Lambda} \right). \quad (6)$$

We see that these transformation rules can be obtained from the commutative ones, just by replacing the ordinary product of smooth functions by the Moyal product, with a suitable product ordering. This allows constructing in simple way invariant quantities.

If the components of the noncommutativity parameter θ are constant, then Lorentz invariance is spoiled. In order to recover it [9, 10, 19] one should change the Moyal star product by the Kontsevich star product $*_K$ [24]. However, as a result of the diffeomorphism invariance, for an even dimensional (symplectic) spacetime X , there exists a local coordinate system (which coincides with Darboux's coordinate system) in which $\theta^{\mu\nu}$ is constant. Therefore, without loss of generality, the Kontsevich product can be reduced to the Moyal one, which will be used from now on.

The fact that the observed world is commutative, means that there must be possible to obtain it from the noncommutative one by taking the limit $\theta \rightarrow 0$. Thus the noncommutative fields \hat{A} are given by a power series expansion on θ , starting from the commutative ones A ,

$$\hat{A} = A + \theta^{\mu\nu} A_{\mu\nu}^{(1)} + \theta^{\mu\nu} \theta^{\rho\sigma} A_{\mu\nu\rho\sigma}^{(2)} + \dots \quad (7)$$

The coefficients of this expansion are determined by the Seiberg-Witten map, which states that the symmetry transformations of (7), given by (4), are induced by the symmetry transformations of the commutative fields (3). In order that these transformations be consistent, the transformation parameter $\widehat{\Lambda}$ must satisfy [18],

$$\delta_\lambda \widehat{\Lambda}(\eta) - \delta_\eta \widehat{\Lambda}(\lambda) - i[\widehat{\Lambda}(\lambda) * \widehat{\Lambda}(\eta)] = \widehat{\Lambda}(-i[\lambda, \eta]). \quad (8)$$

Similarly, the coefficients in Eq. (7) are functions of the commutative fields and their derivatives, and are determined by the requirement that \widehat{A} transforms as (4), [21].

The fact that the noncommutative gauge fields take values in the enveloping algebra, has the consequence that they have a bigger number of components than the commutative ones, unless the enveloping algebra coincides with the Lie algebra of the commutative theory, as is the case of $U(N)$. However, the physical degrees of freedom of the noncommutative fields can be related one to one to the physical degrees of freedom of the commutative fields by the Seiberg-Witten map [6], fact used in references [17, 18, 19, 20, 21] to construct noncommutative gauge theories, in principle for any Lie group.

In order to obtain the Seiberg-Witten map to first order, the noncommutative parameters are first obtained from Eq. (8) [6, 17, 18, 19, 20, 21],

$$\widehat{\Lambda}(\lambda, A) = \lambda + \frac{1}{4}\theta^{\mu\nu} \{\partial_\mu \lambda, A_\nu\} + \mathcal{O}(\theta^2). \quad (9)$$

Then, from Eqs. (4) and (7), the following solution is given

$$\widehat{A}_\mu(A) = A_\mu - \frac{1}{4}\theta^{\rho\sigma} \{A_\rho, \partial_\sigma A_\mu + F_{\sigma\mu}\} + \mathcal{O}(\theta^2), \quad (10)$$

and then for the field strength it turns out that,

$$\widehat{F}_{\mu\nu} = F_{\mu\nu} + \frac{1}{4}\theta^{\rho\sigma} \left(2\{F_{\mu\rho}, F_{\nu\sigma}\} - \{A_\rho, D_\sigma F_{\mu\nu} + \partial_\sigma F_{\mu\nu}\} \right) + \mathcal{O}(\theta^2). \quad (11)$$

The higher coefficients in Eq. (7) can be obtained from the observation that the Seiberg-Witten map preserves the operations of the commutative function algebra, hence the following differential equation can be written [6],

$$\delta\theta^{\mu\nu} \frac{\partial}{\partial\theta^{\mu\nu}} \widehat{A}(\theta) = \delta\theta^{\mu\nu} \widehat{A}_{\mu\nu}^{(1)}(\theta), \quad (12)$$

where $\widehat{A}_{\mu\nu}^{(1)}$ is obtained from $A_{\mu\nu}^{(1)}$ in Eq. (7), by substituting the commutative fields by the noncommutative ones under the $*$ -product.

Let us take the generators T^i of the Lie algebra \mathcal{G} to be hermitian, then the generators T^I of the corresponding enveloping algebra can be chosen to be also hermitian, for instance if they are given by the symmetrized products $: T^{i_1} T^{i_2} \dots T^{i_n} :$. Further, the noncommutative transformation parameters $\widehat{\Lambda}(\lambda, A)$ are functions, whose arguments are matrices. Let us now substitute the matrix products inside $\widehat{\Lambda}(\lambda, A)$, by $MN \rightarrow \frac{1}{2}\{M, N\} - \frac{i}{2}(i[M, N])$, for any two matrices M and N . Hence $\widehat{\Lambda}(\lambda, A)$ can be understood as a function whose nonlinear part of depends polynomially, with complex numerical coefficients, on anticommutators $\{\cdot, \cdot\}$ and commutators $i[\cdot, \cdot]$, of λ, A , and their derivatives. With this understanding, we will continue to write it as $\widehat{\Lambda}(\lambda, A)$, and we have

$$[\widehat{\Lambda}(\lambda, A)]^\dagger = \widehat{\Lambda}^\dagger(\lambda^\dagger, A^\dagger), \quad (13)$$

where $\widehat{\Lambda}^\dagger$ is obtained by complex conjugating the mentioned numerical coefficients.

Let us now consider the hermitian conjugation of the transformation law (3), $(\delta_\lambda A_\mu)^\dagger = \partial_\mu \lambda^\dagger + i[\lambda^\dagger, A_\mu^\dagger]$. From it and (8), taking into account (13), we get,

$$\delta_{\lambda^\dagger} \widehat{\Lambda}^\dagger(\lambda^\dagger, A^\dagger) - \delta_{\eta^\dagger} \widehat{\Lambda}^\dagger(\lambda^\dagger, A^\dagger) - i[\widehat{\Lambda}^\dagger(\lambda^\dagger, A^\dagger) * \widehat{\Lambda}^\dagger(\eta^\dagger, A^\dagger)] = \widehat{\Lambda}^\dagger(-i[\lambda^\dagger, \eta^\dagger], A^\dagger). \quad (14)$$

Comparing this equation with (8), with the mentioned convention, it can be seen that the noncommutative parameters satisfy $[\widehat{\Lambda}(\lambda, A)]^\dagger = \widehat{\Lambda}(\lambda^\dagger, A^\dagger)$. From the transformation law (4), a similar conclusion can be obtained for the noncommutative connection, $[\widehat{A}_\mu(A)]^\dagger = \widehat{A}_\mu(A^\dagger)$, as well for the field strength, $[\widehat{F}_{\mu\nu}(A)]^\dagger = \widehat{F}_{\mu\nu}(A^\dagger)$. By this means, if we have a group with real parameters and hermitian generators, with a hermitian connection, then the noncommutative connection and the noncommutative field strength will be also hermitian.

3. TOPOLOGICAL GRAVITY

In this section we shortly review four-dimensional topological gravity. We start from the following $SO(3, 1)$ invariant action

$$I_{TOP} = \frac{\Theta_G^P}{2\pi} \text{Tr} \int_X R \wedge R + i \frac{\Theta_G^E}{2\pi} \text{Tr} \int_X R \wedge \widetilde{R}, \quad (15)$$

where R is the field strength, corresponding to a $SO(3, 1)$ connection ω

$$R_{\mu\nu}{}^{ab} = \partial_\mu \omega_\nu{}^{ab} - \partial_\nu \omega_\mu{}^{ab} + \omega_\mu{}^{ac} \omega_\nu{}^b{}_c - \omega_\mu{}^{bc} \omega_\nu{}^a{}_c, \quad (16)$$

X is a four dimensional closed pseudo-Riemannian manifold and $\tilde{R}_{\mu\nu}{}^{ab} = -\frac{i}{2}\varepsilon^{ab}{}_{cd}R_{\mu\nu}{}^{cd}$ is the dual with respect to the group. Here the coefficients are the gravitational analogs of the Θ -vacuum in QCD [25, 26, 27].

In this action, the connection satisfies the first Cartan structure equation, which relates it to a given tetrad. This action can be written as the integral of a divergence, and a variation of it with respect to the tetrad vanishes, hence it is metric independent, and therefore topological.

The action (15) arises naturally from the MacDowell-Mansouri type action [28]. A similar construction can be done for $(2+1)$ -dimensional Chern-Simons gravity [29]. Keeping this philosophy in mind, action (15) can be rewritten in terms of the self-dual and anti-self-dual parts, $R^\pm = \frac{1}{2}(R \pm \tilde{R})$, of the Riemann tensor as follows:

$$I_{TOP} = \text{Tr} \int_X (\tau R^+ \wedge R^+ + \bar{\tau} R^- \wedge R^-) = \text{Tr} \int_X (\tau R^+ \wedge R^+ + \bar{\tau} \overline{R^+} \wedge \overline{R^+}), \quad (17)$$

where $\tau = (\frac{1}{2\pi}) (\Theta_G^E + i\Theta_G^P)$, and the bar denotes complex conjugation. In local coordinates on X , this action can be rewritten as

$$I_{TOP} = 2\text{Re} \int_X d^4x \varepsilon^{\mu\nu\rho\sigma} R_{\mu\nu}^+{}^{ab} R_{\rho\sigma ab}, \quad (18)$$

Therefore, it is enough to study the complex action,

$$I = \int_X d^4x \varepsilon^{\mu\nu\rho\sigma} R_{\mu\nu}^+{}^{ab} R_{\rho\sigma ab}. \quad (19)$$

Further, the self-dual Riemann tensor satisfies, $\varepsilon^{ab}{}_{cd}R_{\mu\nu}^+{}^{cd} = 2iR_{\mu\nu}^+{}^{ab}$. This tensor has the useful property that it can be written as a usual Riemann tensor, but in terms of the self-dual components of the spin connection, $\omega_\mu^+{}^{ab} = \frac{1}{2}(\omega_\mu^{ab} - \frac{i}{2}\varepsilon^{ab}{}_{cd}\omega_\mu^{cd})$, as

$$R_{\mu\nu}^+{}^{ab} = \partial_\mu \omega_\nu^+{}^{ab} - \partial_\nu \omega_\mu^+{}^{ab} + \omega_\mu^+{}^{ac} \omega_\nu^+{}^b{}_c - \omega_\mu^+{}^{bc} \omega_\nu^+{}^a{}_c. \quad (20)$$

In this case, the action (18) can be rewritten as,

$$I = \int_X d^4x \varepsilon^{\mu\nu\rho\sigma} [2R_{\mu\nu}{}^{0i}(\omega^+) R_{\rho\sigma 0i}(\omega^+) + R_{\mu\nu}{}^{ij}(\omega^+) R_{\rho\sigma ij}(\omega^+)]. \quad (21)$$

Now, we define $\omega_\mu^i = i\omega_\mu^{+0i}$, from which we obtain, by means of the self-duality properties, $\omega_\mu^{+ij} = -\varepsilon^{ij}{}_k \omega_\mu^k$. Then it turns out that

$$R_{\mu\nu}{}^{0i}(\omega^+) = -i(\partial_\mu \omega_\nu^i - \partial_\nu \omega_\mu^i + 2\varepsilon_{jk}^i \omega_\mu^j \omega_\nu^k) = -i\mathcal{R}_{\mu\nu}^i(\omega) \quad (22)$$

$$R_{\mu\nu}{}^{ij}(\omega^+) = \partial_\mu \omega_\nu^{+ij} - \partial_\nu \omega_\mu^{+ij} - 2(\omega_\mu^i \omega_\nu^j - \omega_\nu^i \omega_\mu^j) = -\varepsilon^{ij}{}_k \mathcal{R}_{\mu\nu}^k(\omega). \quad (23)$$

This amounts to the decomposition $SO(3,1) = SL(2, \mathbf{C}) \times SL(2, \mathbf{C})$, such that ω_μ^i is a complex $SL(2, \mathbf{C})$ connection. If we choose the algebra $\mathfrak{sl}(2, \mathbf{C})$ to satisfy $[T_i, T_j] = 2i\varepsilon_{ij}^k T_k$ and $\text{Tr}(T_i T_j) = 2\delta_{ij}$, then we can write

$$I = \text{Tr} \int_X d^4x \varepsilon^{\mu\nu\rho\sigma} \mathcal{R}_{\mu\nu}(\omega) \mathcal{R}_{\rho\sigma}(\omega), \quad (24)$$

where, $\mathcal{R}_{\mu\nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu - i[\omega_\mu, \omega_\nu]$ is the field strength. This action is invariant under the $SL(2, \mathbf{C})$ transformations, $\delta_\lambda \omega_\mu = \partial_\mu \lambda + i[\lambda, \omega_\mu]$.

In the case of a Riemannian manifold X , the signature and the Euler topological invariants of X , are the real and imaginary parts of (24)

$$\sigma(X) = -\frac{1}{24\pi^2} \text{Re} \text{Tr} \int_X d^4x \varepsilon^{\mu\nu\rho\sigma} \mathcal{R}_{\mu\nu}(\omega) \mathcal{R}_{\rho\sigma}(\omega), \quad (25)$$

$$\chi(X) = \frac{1}{32\pi^2} \text{Im} \text{Tr} \int_X d^4x \varepsilon^{\mu\nu\rho\sigma} \mathcal{R}_{\mu\nu}(\omega) \mathcal{R}_{\rho\sigma}(\omega). \quad (26)$$

4. NONCOMMUTATIVE TOPOLOGICAL GRAVITY

We wish to have a noncommutative formulation of the $SO(3,1)$ action (15). Its first term, can be straightforwardly made noncommutative, in the same way as for usual Yang-Mills theory,

$$\text{Tr} \int_X \hat{R} \wedge \hat{R}. \quad (27)$$

If the $SO(3,1)$ generators are chosen to be hermitian, for example in the spin $\frac{1}{2}$ representation given by $\gamma^{\mu\nu}$, then from the discussion at the end of the second section, it turns out that $\hat{R}_{\mu\nu}$ is hermitian and consequently (27) is real.

Instead, for the second term of (15) such an action cannot be written, because it involves the Levi-Civita symbol, an invariant Lorentz tensor, but which is not invariant under the full enveloping algebra. However, as mentioned at the end of the preceding section, this term can be obtained from Eq. (24).

Thus, in general we will consider as the noncommutative topological action of gravity, the $SL(2, \mathbf{C})$ invariant action,

$$\hat{I} = \text{Tr} \int_X d^4x \varepsilon^{\mu\nu\rho\sigma} \hat{\mathcal{R}}_{\mu\nu} \hat{\mathcal{R}}_{\rho\sigma}, \quad (28)$$

where $\hat{\mathcal{R}}_{\mu\nu} = \partial_\mu \hat{\omega}_\nu - \partial_\nu \hat{\omega}_\mu - i[\hat{\omega}_\mu, \hat{\omega}_\nu]$, is the $SL(2, \mathbf{C})$ noncommutative field strength. This action does not depend on the metric of X . Indeed, as well as the commutative one, it is

given by a divergence,

$$\hat{I} = \text{Tr} \int_X d^4x \varepsilon^{\mu\nu\rho\sigma} \partial_\mu \left(\hat{\omega}_\nu * \partial_\rho \hat{\omega}_\sigma + \frac{2}{3} \hat{\omega}_\nu * \hat{\omega}_\rho * \hat{\omega}_\sigma \right). \quad (29)$$

Thus, a variation of (28) with respect to the noncommutative connection, will vanish identically because of the noncommutative Bianchi identities,

$$\delta_{\hat{\omega}} \hat{I} = 8 \text{Tr} \int \varepsilon^{\mu\nu\rho\sigma} \delta \hat{\omega}_\mu * \hat{D}_\mu \hat{R}_{\rho\sigma} \equiv 0, \quad (30)$$

where \hat{D}_μ is the noncommutative covariant derivative.

Further, from the first Cartan structure equation, the $\text{SO}(3,1)$ connection, and thus its $SL(2, \mathbf{C})$ projection ω_μ^i , can be written in terms of the tetrad and the torsion. Furthermore, from the Seiberg-Witten map, the noncommutative connection can be written as well as $\hat{\omega}(e)$. Therefore, a variation of the action (28) with respect to the tetrad of the action, can be written as

$$\delta_e \hat{I} = 8 \text{Tr} \int \varepsilon^{\mu\nu\rho\sigma} \delta_e \hat{\omega}_\mu(e) * \hat{D}_\mu \hat{R}_{\rho\sigma} \equiv 0, \quad (31)$$

hence it is topological, as the commutative one.

Thus, we see from (29) that, in a θ -power expansion of the action, each one of the resulting terms will be independent of the metrics, as well as they will be given by a divergence. Thus, unless these terms vanish identically, they will be topological. Furthermore, the whole noncommutative action, expressed in terms of the commutative fields by the Seiberg-Witten map, is invariant under the $\text{SO}(3,1)$ transformations. Thus, each term of the expansion will be also invariant. Thus these terms will be topological invariants.

The action (28) is not real, as well as the limiting commutative action. Hence, it is not obvious that the signature (27) will be precisely its real part. In this case we could neither say that $\hat{\chi}(X)$ is given by its imaginary part. In fact we could only say that $\hat{\chi}(X)$ could be obtained from the difference of (28) and (27). However, the real and the imaginary parts of (28) are invariant under $SL(2, \mathbf{C})$ and consequently under $\text{SO}(3,1)$, and thus they are the natural candidates for $\hat{\sigma}(X)$ and $\hat{\chi}(X)$, as in (25) and (26). In order to write down these noncommutative actions as an expansion in θ , we will take as generators for the algebra of $SL(2, \mathbf{C})$, the Pauli matrices. In this case, to second order in θ , the Seiberg-Witten map for the Lie algebra valued commutative field strength $\mathcal{R}_{\mu\nu} = \mathcal{R}_{\mu\nu}^i(\omega) \sigma_i$, is given by

$$\hat{\mathcal{R}}_{\mu\nu} = \mathcal{R}_{\mu\nu} + \theta^{\alpha\beta} \mathcal{R}_{\mu\nu\alpha\beta}^{(1)} + \theta^{\alpha\beta} \theta^{\gamma\delta} \mathcal{R}_{\mu\nu\alpha\beta\gamma\delta}^{(2)} + \dots, \quad (32)$$

where, from Eq. (11) we get,

$$\theta^{\rho\sigma}\mathcal{R}_{\mu\nu\rho\sigma}^{(1)} = \frac{1}{2}\theta^{\rho\sigma} \left[2\mathcal{R}_{\mu\rho}^i \mathcal{R}_{\nu\sigma i} - \omega_\rho^i (\partial_\sigma \mathcal{R}_{\mu\nu i} + D_\sigma \mathcal{R}_{\mu\nu i}) \right] \mathbf{1}, \quad (33)$$

where $\mathbf{1}$ is the unity 2×2 matrix. Further, by means of Eq. (12), we get,

$$\begin{aligned} \theta^{\rho\sigma}\theta^{\tau\theta}\mathcal{R}_{\mu\nu\rho\sigma\tau\theta}^{(2)} &= \frac{1}{4}\theta^{\rho\sigma}\theta^{\tau\theta} \left(\varepsilon_{jk}^i \left[i\partial_\tau \mathcal{R}_{\mu\rho}^j \partial_\theta \mathcal{R}_{\nu\sigma}^k + \partial_\tau \omega_\rho^j \partial_\theta (\partial_\sigma + D_\sigma) \mathcal{R}_{\mu\nu}^k \right] \right. \\ &\quad - \omega_\rho^i \partial_\tau \omega_\sigma^j \partial_\theta \mathcal{R}_{\mu\nu j} + \mathcal{R}_{\mu\rho}^i [2\mathcal{R}_{\nu\tau}^j \mathcal{R}_{\sigma\theta j} - \omega_\tau^j (\partial_\theta + D_\theta) \mathcal{R}_{\nu\sigma j}] \\ &\quad - \mathcal{R}_{\nu\rho}^i [2\mathcal{R}_{\mu\tau}^j \mathcal{R}_{\sigma\theta j} - \omega_\tau^j (\partial_\theta + D_\theta) \mathcal{R}_{\mu\sigma j}] + \frac{1}{2}\omega_\tau^j (\partial_\theta \omega_{\rho j} + \mathcal{R}_{\theta\rho j}) (\partial_\sigma + D_\sigma) \mathcal{R}_{\mu\nu}^i \\ &\quad \left. - 2\omega_\rho^i \{ 2\partial_\sigma \mathcal{R}_{\mu\tau}^j \mathcal{R}_{\nu\theta j} - \partial_\sigma [\omega_\tau^j (\partial_\theta + D_\theta) \mathcal{R}_{\mu\nu j}] \} \right) \sigma_i. \end{aligned} \quad (34)$$

Therefore, to second order in θ , the action (28) will be given by,

$$\widehat{I} = \text{Tr} \int_X d^4x \varepsilon^{\mu\nu\rho\sigma} \left[\mathcal{R}_{\mu\nu} \mathcal{R}_{\rho\sigma} + 2\theta^{\tau\vartheta} \mathcal{R}_{\mu\nu} \mathcal{R}_{\rho\sigma\tau\vartheta}^{(1)} + \theta^{\tau\theta} \theta^{\vartheta\zeta} \left(2\mathcal{R}_{\mu\nu} \mathcal{R}_{\rho\sigma\tau\theta\vartheta\zeta}^{(2)} + \mathcal{R}_{\mu\nu\tau\theta}^{(1)} \mathcal{R}_{\rho\sigma\vartheta\zeta}^{(1)} \right) \right]. \quad (35)$$

Taking into account (33), we get that the first order term is proportional to $\text{Tr}(\sigma_i)$ and thus vanishes identically. Further using (34), we finally get,

$$\begin{aligned} \widehat{I} &= \int_X d^4x \varepsilon^{\mu\nu\rho\sigma} \left\{ 2\mathcal{R}_{\mu\nu}^i \mathcal{R}_{\rho\sigma i} + \frac{1}{4}\theta^{\tau\theta} \theta^{\vartheta\zeta} \left[-\varepsilon_{ijk} R_{\mu\nu}^i \left[\partial_\vartheta R_{\rho\tau}^j \partial_\zeta R_{\sigma\theta}^k - \partial_\vartheta \omega_\tau^j \partial_\zeta (\partial_\theta + D_\theta) R_{\rho\sigma}^k \right] \right. \right. \\ &\quad + [R_{\mu\tau}^i R_{\nu\theta i} - \frac{1}{2}\omega_\tau^i (\partial_\theta + D_\theta) R_{i\mu\nu}] [R_{\rho\vartheta}^j R_{\sigma\zeta j} - \frac{1}{2}\omega_\vartheta^j (\partial_\zeta + D_\zeta) R_{\rho\sigma j}] \\ &\quad + R_{\mu\nu}^i \{ R_{i\sigma\theta} [2R_{\rho\vartheta}^j R_{\tau\zeta j} - \omega_\vartheta^j (\partial_\zeta + D_\zeta) R_{\rho\tau j}] + \frac{1}{4}(\partial_\theta + D_\theta) R_{\rho\sigma i} \omega_\vartheta^j (\partial_\zeta \omega_{\tau j} + R_{\zeta\tau j}) \\ &\quad \left. \left. + \omega_{\theta i} [\partial_\tau (R_{\rho\vartheta}^j R_{\sigma\zeta j}) - \frac{1}{2}\partial_\tau \omega_\vartheta^j (\partial_\zeta + D_\zeta) R_{\rho\sigma j}] \right\} - \frac{1}{2} R_{\mu\nu}^i \omega_{\tau i} \partial_\vartheta \omega_\theta^j \partial_\zeta R_{\rho\sigma j} \right] \Big\}, \end{aligned} \quad (36)$$

where the second order correction does not identically vanish.

Similarly to the second order term (34), the third order term for $\widehat{\mathcal{R}}$ can be computed by means of Eq. (12). The result is given by a rather long expression, which however is proportional to the unity matrix $\mathbf{1}$, like (33). Thus the third order term in (35), given by

$$2\theta^{\tau_1\theta_1}\theta^{\tau_2\theta_2}\theta^{\tau_3\theta_3} \text{Tr} \int_X \varepsilon^{\mu\nu\rho\sigma} \left(\mathcal{R}_{\mu\nu} \mathcal{R}_{\rho\sigma\tau_1\theta_1\tau_2\theta_2\tau_3\theta_3}^{(3)} + \mathcal{R}_{\mu\nu\tau_1\theta_1}^{(1)} \mathcal{R}_{\rho\sigma\tau_2\theta_2\tau_3\theta_3}^{(2)} \right), \quad (37)$$

vanishes identically, because $\mathcal{R}^{(2)}$ is proportional to σ_i . Thus, (36) is valid to third order. In fact, it seems that all its odd order terms vanish.

5. TOWARDS NONCOMMUTATIVE GRAVITATIONAL INSTANTONS AND ANOMALIES

5.1. Towards Noncommutative Gravitational Instantons

In the Euclidean signature, the action (15), with local Lorentz group $SO(4)$, is proportional to a linear combination of integer valued topological invariants, the Euler $\chi(X)$ and the signature $\sigma(X)$, which characterize the gravitational instantons. In fact, $\sigma(X)$ and $\chi(X)$ are the analogue of the instanton number k of $SU(2)$ -Yang-Mills instantons, which is a manifestation of the gauge group topology, through $k \in \pi_3(SU(2))$. These topological invariants χ and σ , should of course include the corresponding boundary and η -invariant terms. Gravitational instantons are finite action solutions of the self-dual Einstein equations, which are asymptotically Euclidean [30], or asymptotically locally Euclidean (ALE) [31], at infinity (for a review, see [32]). Then one would ask about the possibility to get gravitational instanton solutions in noncommutative gravity. The first natural step would be to analyze the positive action conjecture [33], in the context of noncommutative gravity, although it would require a more complete version of noncommutative gravity. However, it is possible to give some generic arguments, and we will focus on the description of the global aspects, by analyzing invariants χ and σ in the noncommutative context. In order to do that, we concentrate in the spin connection dependence, leaving the explicit metrics for later analysis.

In the previous section, from explicit computations of the noncommutative corrections (in the noncommutative parameter θ) of the topological invariants (see Eq. (36)), we got that they do not vanish at $\mathcal{O}(\theta^2)$, hence the classical topological invariants are clearly modified. Thus, the use of the Seiberg-Witten map for the Lorentz group leads to essentially modified invariants $\hat{\chi}$ and $\hat{\sigma}$, which would characterize ‘noncommutative gravitational instantons’. Further, the corresponding deformed equation under the Seiberg-Witten map, $\hat{R}_{\mu\nu}^+ = 0$, does admit an expansion in θ with the term at the zero order being $R_{\mu\nu}^+$. Thus these corrections should be associated to the θ -corrections of the self-duality equation $R_{\mu\nu}^+ = 0$. Furthermore, we could expect for the gravitational instantons similar effects as for the case of Yang-Mills instantons [6, 34], where the singularities of moduli space are resolved by the noncommutative deformations.

We already know from models of the minisuperspace in quantum cosmology, that noncommutative gravity leads to a version of noncommutative minisuperspace [35]. Thus, one would expect some new physical effects from the moduli space of metrics of a noncommutative gravity theory, which may help to resolve spacetime singularities.

5.2. Comments on Gravitational Anomalies in Noncommutative Spaces

- *A Brief Survey on Gravitational Anomalies*

The study of topological invariants, leads us also to other nontrivial topological effects, like the anomalies, in our gravitational case. Gravitational anomalies, as well as gauge anomalies, are classified in local and global anomalies. In this paper we will mainly focus on local anomalies, whereas global anomalies will be mentioned as reference for future work.

Local anomalies are associated to the lack of invariance of the quantum one-loop effective action, under infinitesimal local transformations. There are different types of local gravitational anomalies, depending on the type of transformations, like the Lorentz (or automorphisms) anomaly, and the diffeomorphisms anomaly.

Let \mathcal{G}_0^L be the group of vertical automorphisms of the frame bundle over the spacetime X . In a local trivialization, the frame bundle \mathcal{G}_0^L can be identified with the set of continuous maps from X to $SO(4)$, which approach to the identity at infinity, i.e. $\mathcal{G}_0^L \equiv \text{Map}_0(X, SO(4)) \equiv \{g : X \rightarrow SO(4), g \text{ continuous}\}$. Let \mathcal{W} be the space of gauge field configurations, which consists of all spin connections $\omega_\mu^{ab}(x)$ with appropriate boundary conditions, and let $\mathcal{B} = \mathcal{W}/\mathcal{G}_0^L$. The automorphisms group \mathcal{G}_0^L acts on \mathcal{W} in such a way that one can construct the gauge bundle: $\mathcal{G}_0^L \rightarrow \mathcal{W} \xrightarrow{\pi} \mathcal{B}$. In spacetimes X of $n = \dim X = 2m$ dimensions, the existence of the local Lorentz gravitational anomaly is associated to the non-triviality of the non-torsion part of the homotopy of \mathcal{B} , i.e. $\pi_2(\mathcal{B}) \cong \pi_1(\mathcal{G}_0^L) = \pi_{2m+1}(SO(2m)) \neq 1$. For instance for $X = S^4$, we get the pure topological torsion $\pi_1(\mathcal{G}_0^L) \cong \pi_5(SO(4)) = \pi_5(SU(2) \times SU(2)) = \mathbf{Z}_2 \oplus \mathbf{Z}_2$. Thus, in four dimensions there is no local Lorentz anomaly. However, in $n = 4k + 2$ dimensions, for $k = 0, 1, \dots$, it certainly exists.

For local diffeomorphisms transformations, the moduli space involves a richer phase space structure, given by the quotient space of the Teichmüller space, and the mapping class group. These anomalies can exist only for $n = 4k + 2$ dimensions for $k = 0, 1, 2, \dots$. However, mixed local Lorentz and diffeomorphism anomalies can exist in $2k + 2$ dimensions [36].

Global gravitational Lorentz anomalies arise from the fact that Lorentz transformations are disconnected, which is related to the nontrivial topology of the group $\mathcal{G}_\infty^L = \mathcal{G}^L/\mathcal{G}_0^L$, where \mathcal{G}^L is the set of local Lorentz transformations which have a limit at infinity. In particular for $X = S^4$, $\pi_0(\mathcal{G}_\infty^L) \cong \pi_4(SO(4)) = \pi_4(SU(2) \times SU(2)) = \mathbf{Z}_2 \oplus \mathbf{Z}_2$, and a nontrivial global Lorentz anomaly arises. Similarly, the global gravitational diffeomorphisms anomalies are related to the disconnectedness of the mapping class group Γ_∞^+ , i.e. $\pi_0(\Gamma_\infty^+) \neq 1$ [37].

• *Noncommutative Local Lorentz Anomalies*

Let us turn to the noncommutative side. The noncommutative version of the Lorentz group will be denoted by $\widehat{SO(4)}$, and it is defined in terms of some suitable operator algebra on a *real* Hilbert space. Here and in the following, unless otherwise stated, the noncommutative spaces and groups corresponding to the ones in the preceding section, will be denoted by hatted ones. Following [38], we propose that $\widehat{SO(4)}$ will be given by the set of compact orthogonal operators $\mathbf{O}_{cpt}(\mathcal{H})$, defined on the separable real Hilbert space \mathcal{H} . The compactness property avoids the Kuiper theorem, which states that the set of pure orthogonal operators $\mathbf{O}(\mathcal{H})$ has trivial homotopy groups [39]. However, the restriction to subalgebras of normed orthogonal operators $\mathbf{O}_p(\mathcal{H}) = \{\alpha \mid \alpha = \mathbf{1} + K\}$ has very important consequences. Here K stands for compact, finite rank, trace class and Hilbert-Schmidt operator. By a mathematical result [40], the family of normed operator algebras $(\mathbf{O}_p(\mathcal{H}), \|\cdot\|_p)$, with the L^p -norm given by $\|D\|_p = (\text{Tr}|D|^p)^{1/p}$, together with the set $(\mathbf{O}_{cpt}(\mathcal{H}), \|\cdot\|_\infty)$, have exactly the same stable homotopy groups as $SO(\infty)$ (defined through the Bott periodicity theorem). Further, the stable homotopy groups of $SO(\infty)$, $\pi_j(SO(\infty))$, are given by \mathbf{Z}_2 for $j = 0$, \mathbf{Z}_2 for $j = 1$, \mathbf{Z} for $j = 3$, and 1 otherwise. Also these groups have Bott periodicity mod 8, i.e. $\pi_n(SO(\infty)) = \pi_{n+8}(SO(\infty))$. Thus, the stable homotopy groups of $\widehat{SO(4)} = \mathbf{O}_{cpt}(\mathcal{H})$ are in general nontrivial, and new topological effects in noncommutative gravity theories are possible.

Let us turn now to the noncommutative analogue of the local Lorentz anomaly. It is determined by the nontrivial non-torsion part of homotopy groups of a suitable noncommutative version of the Lorentz group $\widehat{\mathcal{G}}_0^L$, which could be defined as the set $\widehat{\mathcal{G}}_0^L \equiv \text{Map}_0(X, \mathbf{O}_{cpt}(\mathcal{H}))$. The noncommutative local Lorentz anomaly is detected by the homotopy group $\pi_2(\widehat{\mathcal{B}}) = \pi_1(\widehat{\mathcal{G}}_0^L) = \pi_j(\mathbf{O}_{cpt}(\mathcal{H})) \neq 1$ for $j = 0, 1, 3 \bmod 8$. For $j = 0, 1$ we have $\pi_j(\mathbf{O}_{cpt}(\mathcal{H})) = \mathbf{Z}_2$, while for $j = 3$, $\pi_j(\mathbf{O}_{cpt}(\mathcal{H})) = \mathbf{Z}$. Thus for $j = 3$ a non-torsion part is detected, and

therefore the existence of a local Lorentz anomaly.

Finally, in the global perspective, the Seiberg-Witten map can be regarded as a map $SW : \mathcal{B} \rightarrow \widehat{\mathcal{B}}$, which preserves the infinitesimal Lorentz transformation (the gauge equivalence relation), and thus the locally Lorentz invariant observables of the theory. The Seiberg-Witten map is not well defined globally since both spaces \mathcal{B} and $\widehat{\mathcal{B}}$ are different, and their corresponding topologies can be different as well. However, in some specific cases the operator representation of the Seiberg-Witten map is quite useful to define the Seiberg-Witten map globally [41].

6. CONCLUDING REMARKS

In this paper, we propose a noncommutative version for topological gravity with quadratic actions. We start by the complex action (28), in terms of the self-dual and antiself-dual connections, and which contains both the signature and the Euler topological invariants (for a Riemannian manifold). This action is then written as a $SL(2, \mathbb{C})$ action, whose noncommutative counterpart can be obtained in the same way as in the Yang-Mills case, by means of the Seiberg-Witten map. We compute this action up to third θ -order, and we obtain that the first and the third order vanish, but the second order is different from zero. The action to this order is given by (36). It seems that all odd θ -orders vanish identically.

The noncommutative signature and the Euler topological invariants are given by the real and imaginary parts of (28). For a Riemannian manifold, these topological invariants characterize gravitational instantons. Thus the study of noncommutative topological invariants should allow us, through the Seiberg-Witten map, to deform gravitational instantons into noncommutative versions for them. In order to make explicit computations, specific gravitational (noncommutative) metrics have to be chosen. In this context, it would be very interesting to give a noncommutative formulation for dynamical gravity, following the lines of this work. This analysis will be reported in a forthcoming paper [42].

Similarly to the gauge theories case, we propose a definition of noncommutative local gravitational Lorentz anomaly, by a suitable definition of the noncommutative Lorentz group $\widehat{SO(4)}$ in compact spacetime of Euclidean signature. The application of these ideas to the diffeomorphism transformations connected to the identity might predict new nontrivial non-

commutative gravitational effects, which should be computed explicitly as a noncommutative correction to the gravitational contribution to the chiral anomaly. The usual gravitational correction was computed for the standard commutative case in Refs. [36, 43]. Moreover, this effect can also be regarded as a noncommutative gravitational correction of the local chiral anomaly in noncommutative gauge theory. This latter case of the pure noncommutative gauge field was discussed recently in Refs. [44]. It would be very interesting to pursue this way and compare with the results given recently by Perrot [45].

Regarding noncommutative global Lorentz anomalies, in order to understand them, we would need to specify the connected components of the corresponding group $\widehat{\mathcal{G}}_\infty^L$. In this case one would have to compute $\pi_1(\widehat{\mathcal{W}}/\widehat{\mathcal{G}}_\infty^L) = \pi_0(\widehat{\mathcal{G}}_\infty^L) \neq 1$. Of course a suitable operator definition of $\widehat{\mathcal{G}}_\infty^L$ is necessary like in the case of the local Lorentz anomaly. This is a difficult open problem.

Finally, the ALE gravitational instantons is an important case of gravitational instantons, which can be obtained as smooth resolutions of **A-D-E** orbifold singularities \mathbf{C}^2/Γ , with Γ being an **A-D-E** finite subgroup of $SU(2)$. These gravitational instantons are classified through the Kronheimer construction [46], which is the analogue construction to the ADHM construction of Yang-Mills instantons. There is a proposal to extend the ADHM construction to the noncommutative case [34]. Thus, it would be interesting to give the noncommutative analogue of the Kronheimer construction of ALE instantons.

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